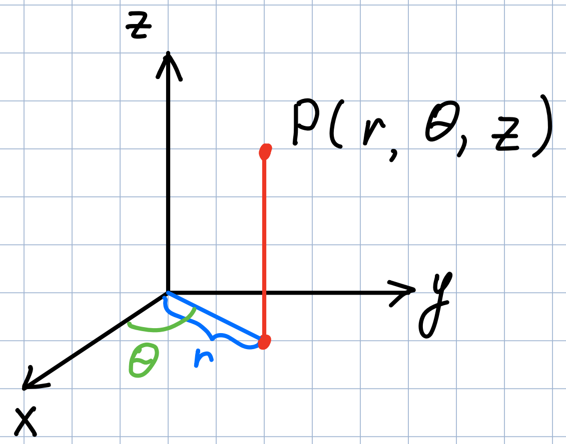
Last time: Triple integrals in cylindrical coordinates



$\underbrace{r, \theta}_{\text{polar coord. of proj. of } P \text{ onto } xy\text{-plane}}, z$ - cylindrical coordinates of a point P

$$x = r\cos\theta, \quad y = r\sin\theta \implies x^2 + y^2 = r^2$$

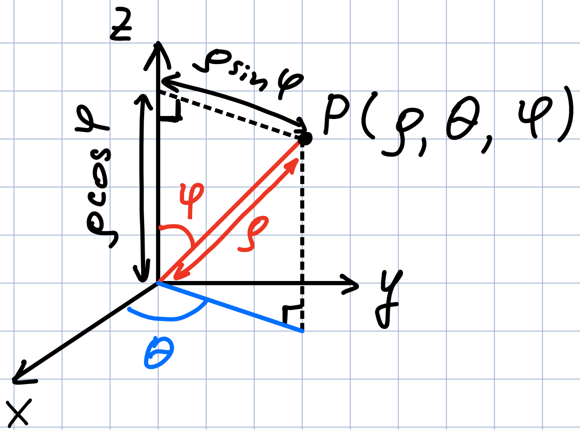
$$z = z$$

For a solid

$$E: \left\{ \begin{array}{c} \alpha \le \theta \le \beta \\ h_1(\theta) \le r \le h_2(\theta) \\ g_1(r,\theta) \le z \le g_2(r,\theta) \end{array} \right\}$$

$$\iiint_E f(x,y,z)\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)\, r \, dz\, dr\, d\theta$$

Triple integrals in spherical coordinates



$\rho = |OP|$

θ - same as in cylindrical

φ - angle between $\vec{OP}$ and z-axis

$\rho \geq 0, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \varphi \leq \pi$

$z = \rho \cos\varphi$

$x = \rho \sin\varphi \cos\theta$

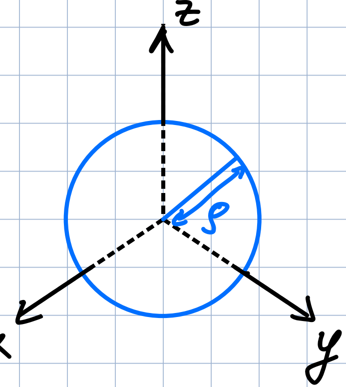
$y = \rho \sin\varphi \sin\theta$

$r = \rho \sin\varphi$

↑ r of cylindrical

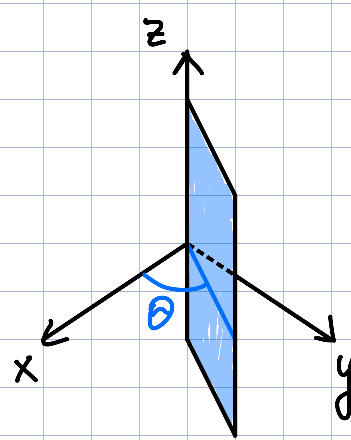
Rmk: $\rho^2 = x^2 + y^2 + z^2$

$\rho = \text{const}$



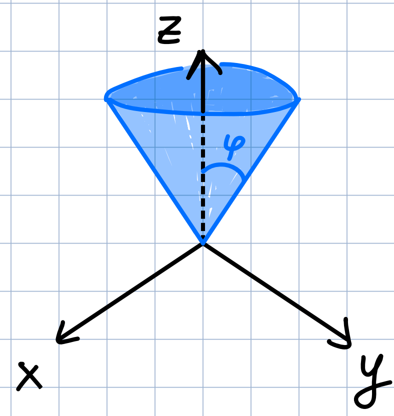
A sphere

$\theta = \text{const}$



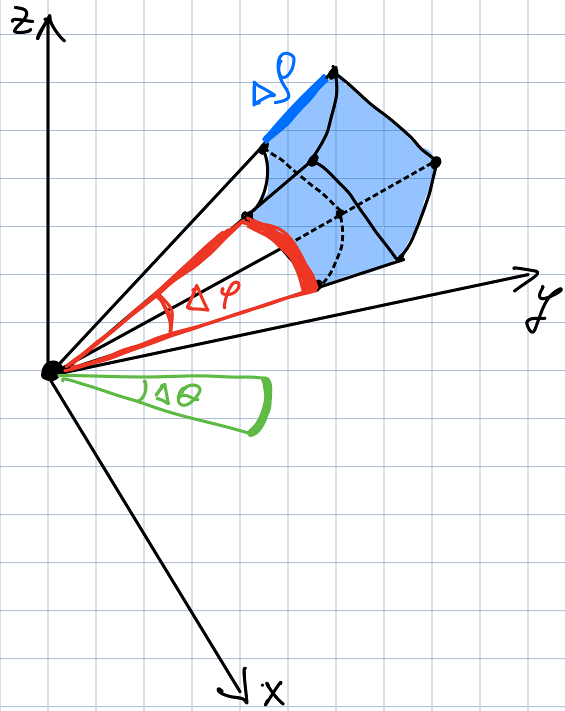
A half-plane

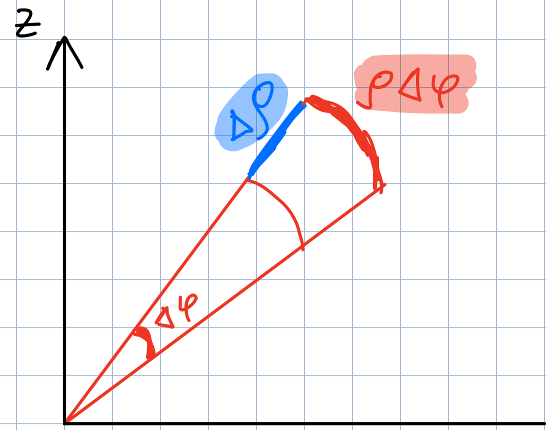
$\varphi = \text{const}$

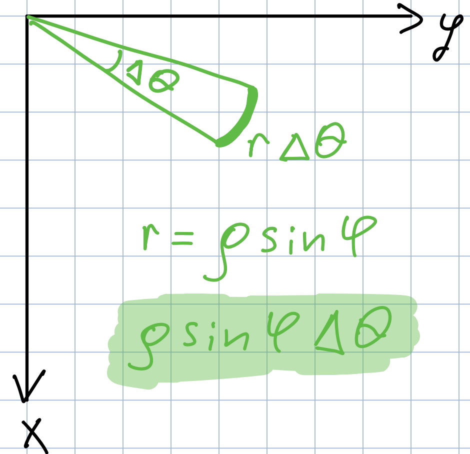


A cone

$$E = \left\{ (\rho, \theta, \varphi) \;\middle|\; \begin{array}{c} c \le \varphi \le d \\ \alpha \le \theta \le \beta \\ a \le \rho \le b \end{array} \right\} \text{ - spherical wedge}$$







$$\Delta V = \rho^2 \sin\varphi \, \Delta\varphi \, \Delta\theta \, \Delta\rho$$

$$\iiint_E F(x,y,z)\, dV = \int_c^d \int_\alpha^\beta \int_a^b F(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\, \rho^2\sin\varphi \, d\rho\, d\theta\, d\varphi$$

Rmk: Can extend this to more general spherical regions

$E: \; c \le \varphi \le d, \quad \alpha \le \theta \le \beta, \quad g_1(\theta,\varphi) \le \rho \le g_2(\theta,\varphi)$

When boundary of E is formed of cones and spheres – use spherical coordinates.

Ex: Find $\iiint_E e^{(x^2+y^2+z^2)^{3/2}}\, dV$, where $E = \{(x,y,z) \mid x^2+y^2+z^2 \le 1\}$ – unit ball.

Sol: $E = \left\{ (\rho, \theta, \varphi) \;\middle|\; \begin{array}{c} 0 \le \varphi \le \pi \\ 0 \le \theta \le 2\pi \\ 0 \le \rho \le 1 \end{array} \right\}$

$$\iiint_E e^{(x^2+y^2+z^2)^{3/2}}\, dV \underset{x^2+y^2+z^2=\rho^2}{=} \int_0^{\pi}\int_0^{2\pi}\int_0^{1} e^{(\rho^2)^{3/2}} \rho^2 \sin\varphi \, d\rho\, d\theta\, d\varphi$$

$$= \int_0^{\pi}\int_0^{2\pi}\underbrace{\int_0^{1} e^{\rho^3} \rho^2 \sin\varphi \, d\rho}_{\frac{1}{3} e^{\rho^3} \sin\varphi \Big|_{\rho=0}^{\rho=1} = \frac{1}{3}\sin\varphi (e-1)}\, d\theta\, d\varphi = \int_0^{\pi}\int_0^{2\pi} \frac{1}{3}(e-1)\sin\varphi\, d\theta\, d\varphi$$

$$= \frac{2\pi}{3}(e-1)\int_0^{\pi} \sin\varphi\, d\varphi = \frac{4\pi}{3}(e-1)$$

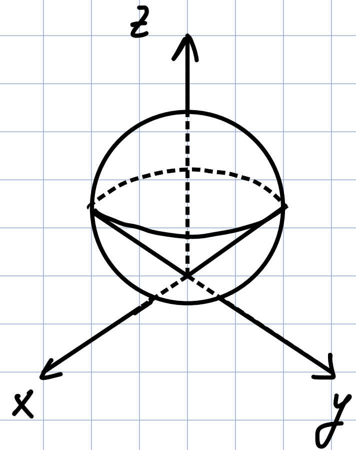
Rmk: In rectangular:

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} e^{(x^2+y^2+z^2)^{3/2}}\, dz\, dy\, dx$$

HARD!

Ex: E - solid above the cone $z=\sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=z$. Find the volume of E.

Sol: $x^2+y^2+z^2-z=0 \iff x^2+y^2+\underbrace{z^2-z+\frac{1}{4}}_{(z-\frac{1}{2})^2}=\frac{1}{4}$



sphere: $\underbrace{x^2+y^2+z^2}_{\rho^2}=\underbrace{z}_{\rho\cos\varphi} \implies \rho=\cos\varphi$

cone: $z=\sqrt{\underbrace{x^2+y^2}_{\rho^2\sin^2\varphi(\cos^2\theta+\sin^2\theta)}} \implies \rho\cos\varphi=\rho\sin\varphi$

$\implies \cos\varphi=\sin\varphi$

$\implies \varphi=\frac{\pi}{4}$

So

$$E=\left\{(\rho,\theta,\varphi)\ \middle|\ \begin{array}{l} 0\le\varphi\le\frac{\pi}{4} \\ 0\le\theta\le 2\pi \\ 0\le\rho\le\cos\varphi \end{array}\right\}$$

$$\text{Volume}(E)=\iiint_E 1\,dV=\int_0^{\frac{\pi}{4}}\int_0^{2\pi}\underbrace{\int_0^{\cos\varphi}\rho^2\sin\varphi\,d\rho}_{\frac{1}{3}\cos^3\varphi\sin\varphi}\,d\theta\,d\varphi=\frac{2\pi}{3}\underbrace{\int_0^{\frac{\pi}{4}}\cos^3\varphi\sin\varphi\,d\varphi}_{-\frac{1}{4}\cos^4\varphi\Big|_{\varphi=0}^{\varphi=\frac{\pi}{4}}}$$

$$=\frac{2\pi}{3}\,\frac{1}{4}\left(1-\frac{1}{4}\right)=\frac{\pi}{8}$$